Math Circles - Elementary Number Theory - Fall 2023

Exercises

GCD

1. Prove the following statements:
   (a) Prove that if $a \mid b$, then $a \mid xb$ for all integers x.
   (b) Prove that if $a \mid b$ and $a \mid c$, then $a \mid b \pm c$.
   (c) Prove that if $a \mid b$ and $a \mid c$, then $a \mid xb \pm cy$ for all integers x and y.
   (d) Prove that if $a \mid b$ and $b \mid c$, then $a \mid c$.
2. Show that if $k \mid mn$ but $k \nmid m$, then $k \mid n$.
3. Compute the following using the Euclidean Algorithm:
   (a) $\gcd(18, 24)$
   (b) $\gcd(78, 320)$
   (c) $\gcd(191, 443)$
4. Let $d = \gcd(a, b)$. Prove that if $c \mid a$ and $c \mid b$, then $c \mid d$.
5. Let d be a common factor of a and b such that all common factors of a and b divide d. Prove that $d = \gcd(a, b)$.

Prime Numbers

1. Write the prime factorizations of the following numbers:
   (a) 2310
   (b) 2048
   (c) 2039
2. Prove that there are infinitely many primes.
3. Suppose a and b are relatively prime. Prove that there exist integers x and y such that $ax + by = 1$.
4. Compute the following:
   - $\Phi(27)$
   - $\Phi(37)$
   - $\Phi(210)$
5. Prove that, for all n, we have that $\sum_{d|n} \Phi(n) = n$.